Limit theorems for stationary distributions of birth-and-death Markov chains

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Consider a discrete-time Markov chain $\zeta_N^t,\ t \geq 1$, assuming the N values $0, \frac{1}{N}, \dots, 1 - \frac{1}{N}$. Let

$$\mathbf{P}\{\zeta_N^{t+1} = \frac{i+1}{N} \mid \zeta_N^t = \frac{i}{N+1}\} = p_i^{(N)},$$

$$\mathbf{P}\{\zeta_N^{t+1} = \frac{i-1}{N} \mid \zeta_N^t = \frac{i}{N+1}\} = q_i^{(N)},$$

with $q_i^{(N)} = 1 - p_i^{(N)}$. Set D_N for the stationary distribution of the chain, assuming that it exists.

Define

$$\Phi_N(x) = \begin{cases} 0 \text{ for } 0 \le x < \frac{1}{N}, \\ -\frac{1}{N}\sum_{j=1}^{i} \ln \frac{p_{j-1}^{(N)}}{q_j^{(N)}} \text{ for } \frac{i}{N} \le x < \frac{i+1}{N},\ 1 \le i \le N-2, \\ -\frac{1}{N}\sum_{j=1}^{N-1} \ln \frac{p_{j-1}^{(N)}}{q_j^{(N)}} \text{ for } 1-\frac{1}{N} \le x \le 1. \end{cases}$$

Assume that there is a continuous in $[0, 1]$ function $\Phi(x)$ such that

$$\sup_{x\in[0,1]} \mid \Phi_N(x) - \Phi(x) \mid = o(\frac{1}{N}).$$

Set $\Phi^* = \min_{x\in[0,1]} \Phi(x)$ and $X^* = \{x \in [0,1] : \Phi(x) = \Phi^*\}$. The set X^* is supposed to have a finite number of connected components. Then all weak limit points of D_N belong to X^* as $N \to \infty$.

To characterize the limits, assume that

$$\Phi(x) = \Phi^* + \psi_i(x - a_i) \quad \text{in a neighborhood of } a_i$$

and

$$\Phi(x) = \Phi^* + \Psi_i(\min[x - b_i, \max(0, x - c_j)]) \qquad \text{in a neighborhood of } [b_j, c_j].$$

Here $a_i \in X^*$ and $[b_j, c_j] \in X^*$. $\psi_i(0) = \Psi_j(0) = 0$, these growth functions decrease for negative values of the argument and they increase for positive values of the argument.

Let all I components of X^* be sigleton and let for every i there be pairs $\gamma_i^+ \geq 2$, α_i^+ and $\gamma_i^- \geq 2$, α_i^- such that

$$\psi_i(u) \approx \alpha_i^+ u^{\gamma_i^+} \text{for } u > 0 \quad \text{and} \quad \psi_i(u) \approx \alpha_i^- (-u)^{\gamma_i^-} \text{for } u < 0.$$

Then the limit distribution of D_N concentrates on the components with the largest $\gamma(i) = \max(\gamma_i^+, \gamma_i^-)$. If $\gamma(s) = \max_{i \in I} \gamma(i)$, then the probability assigned to a_s is proportional to

$$\frac{\Gamma(1/\gamma(s))}{\gamma(s)\alpha(s)^{1/\gamma(s)}}.$$

Here

$$\Gamma(z) = \int_0^\infty \exp(-x) x^{z-1} dx \quad \text{and} \quad \alpha(s) = \begin{cases} \alpha_s^+ \text{ for } \gamma_s^+ > \gamma_s^-, \\ \alpha_s^- \text{ for } \gamma_s^- > \gamma_s^+. \end{cases}$$

When every $\gamma(i)$ equals to 2, the probability assigned to a_s becomes

$$\frac{1/\sqrt{\alpha_s^+} + 1/\sqrt{\alpha_s^-}}{\sum_{i=1}^{I}(1/\sqrt{\alpha_i^+} + 1/\sqrt{\alpha_i^-})}.$$

If there are intervals along with singletons among the connected components of X^*, then the limit of D_N concentrates only of the intervals. The corresponding distribution has a density with respect to the Lebesgue measure.

The study was motivated by the following approach widely used in sociology, economics and population biology.

Set $\Delta\zeta_N^t = \zeta_N^{t+1} - \zeta_N^t$, then

$$\mathbf{E}(\Delta\zeta_N^t \mid \zeta_N^t = \frac{i}{N}) = \frac{1}{N}(p_i^{(N)} - q_i^{(N)}) \text{ and } \mathbf{E}[(\Delta\zeta_N^t)^2 \mid \zeta_N^t = \frac{i}{N}] = \frac{1}{N^2}(p_i^{(N)} + q_i^{(N)}).$$

When $p_i^N \approx f(\frac{i}{N})$ for some continuous function f (then $q_i^N \approx 1 - f(\frac{i}{N})$), it is intuitive to try to relate the limit properties of the process in hands with asymptotic behavior of the following ordinary differential equation

$$\frac{dx}{dt} = 2f(x) - 1.$$

The equilibria of the ODE obtain from the equation $2f(x) = 1$. The Gibbs potential $\Phi(x)$ in this case reads

$$\Phi(x) = \int_0^x \ln \frac{f(z)}{1 - f(z)} dz$$

and its global minimum points satisfy the equation $\ln \frac{f(x)}{1-f(x)} = 0$ or, again, $2f(x) = 1$. This is a subset of the set of locally asymptotically stable equilibria. In other words, the intuition is misleading in this case.