Law invariant convex risk measures on $\mathbb{R}^d$

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The classical one-dimensional case

We extend the following theorems on risk measures, which are well-known for the one-dimensional case, to the d-dimensional case.

We work on a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Reminder

A function $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is called *law invariant* if $X \sim Y$, i. e. $law(X) = law(Y)$, implies that $\varrho(X) = \varrho(Y)$.

For $F \in L^1_+(\Omega, \mathcal{F}, \mathbb{P})$ normalized by $\mathbb{E}[F] = 1$, we define the law invariant risk measure $\varrho_F : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ by

$$\varrho_F(X) = \sup_{\tilde{X} \sim X} \mathbb{E}[-\tilde{X} F] = \sup_{\tilde{F} \sim F} \mathbb{E}[-X \tilde{F}].$$

The measures ϱ_F have the following co-monotonicity property.

Definition

A risk measure $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is called co-monotone if, for co-monotone random variables X, Y, we have

$$\varrho(X + Y) = \varrho(X) + \varrho(Y).$$

Recall that X is co-monotone to Y iff $[X(\omega) - X(\omega')] \cdot [Y(\omega) - Y(\omega')] \geq 0$, for $\mathbb{P} \otimes \mathbb{P}$ almost all $(\omega, \omega') \in \Omega \times \Omega$. We write $X \sim_c Y$.

Theorem A (Kusuoka, 2001)

For a law invariant convex risk measure $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ t.f.a.e.

(i) ϱ is co-monotone.

(ii) There is $F \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $0 \le \alpha \le 1$ s.t.

$$\begin{aligned} \varrho(X) &= \alpha \varrho_F(X) + (1-\alpha) \operatorname{ess\,sup}(-X) \\ &=: \alpha \varrho_F(X) + (1-\alpha)\varrho^\infty(X). \end{aligned}$$

(iii) ϱ is *strongly cohorent*, i.e.

$$\varrho(X) + \varrho(Y) = \sup_{\tilde{X} \sim X, \tilde{Y} \sim Y} \varrho(\tilde{X} + \tilde{Y}).$$

The equivalence (i) $\Leftrightarrow$ (ii) is due to Kusuoka.
Property (iii) is easily seen to be equivalent to (i) in the one-dimensional setting.
While it is – a priori – not clear how to extend the notion (i) of *co-monotonicity* to the vector-valued setting, the notion (iii) of *strong coherence* extends to the vector-valued case on an obvious way. This is the reason why Ekeland-Galichon-Henry (2009) introduced this notion (in the vector valued setting).

Here is the second theorem which we state for the one-dimensional case, and later extend to the d-dimensional one.

Theorem B (Kusuoka,2001)

Let $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a law invariant convex risk measure. Then there is a function $v : [0,1] \times \mathcal{P} \to [0,\infty]$ such that

$$\varrho(X) = \max_{(\alpha,F)\in[0,1]\times\mathcal{P}} \left\{ \alpha\varrho_F(X) + (1-\alpha)\varrho^\infty(X) - v(\alpha,F) \right\}.$$

The d-dimensional case

Definition

A convex risk measure on $\mathbb{R}^d$ is a function
$\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R}$ s.t.
(i) (normalization) $\quad \varrho(0) = 0,$,
(ii) (monotonicity) $\quad X \geq Y \Rightarrow \varrho(X) \leq \varrho(Y),$
(iii) (cash invariance) $\quad \varrho(X + m\mathbb{1}) = \varrho(X) - m, \qquad$ for $m \in \mathbb{R}$,
(iv) (convexity)
$\varrho(\alpha X + (1-\alpha)Y) \leq \alpha\varrho(X) + (1-\alpha)\varrho(Y), \qquad$ for $0 < \alpha < 1$.

We call ϱ *coherent* if, in addition,
(v) (positive homogeneity) $\quad \varrho(\lambda X) = \lambda\varrho(X), \qquad$ for $\lambda \geq 0$.

On the standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we now denote by $\mathcal{P}$ the subset of $L^1_+(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ of normalized functions F taking their values in $\mathbb{R}^d_+$,

$$\mathcal{P} = \left\{ F = (F_i)_{i=1}^d \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) : F_i \geq 0, \mathbb{E}\left[\sum_{i=1}^d F_i\right] = 1 \right\}.$$

Definition (Rüschendorf)

For given $F \in \mathcal{P}$ we define ϱ_F, the *maximal correlation risk measure in the direction F*, by

$$\varrho_F(X) = \sup_{\tilde{X} \sim X} \mathbb{E}\left[-(\tilde{X}, F)\right] = \sup_{\tilde{F} \sim F} \mathbb{E}\left[-(X, \tilde{F})\right].$$

Note that ϱ_F only depends on the law of F.

Proposition (Rüschendorf)

A coherent (resp. convex) risk measure $\varrho : L_d^\infty \to \mathbb{R}$ is *law invariant* if and only if it can be represented as

$$\varrho(X) = \sup_{F \in C} \varrho_F(X)$$

$$\text{resp. } \varrho(X) = \sup_{F \in C} \{\varrho_F(x) - v(F)\}$$

where C is a subset of $\mathcal{P}$ and $v : C \to \mathbb{R}_+$ a non-negative function defined on C.

Apart from the risk measures ϱ_F, where $F \in \mathcal{P}$, a second type of risk measures will play a special role. It generalizes the maximal loss measure from the one- to the d-dimensional case.

Definition

For $\xi \in S^d$, where

$$S^d := \left\{ \xi \in \mathbb{R}^d : \xi_i \geq 0, \sum_{i=1}^{d} \xi_i = 1 \right\},$$

we define the *maximal loss measure in the direction* ξ by

$$\varrho_\xi^\infty(X) = ess \sup \left\{ -\sum_{i=1}^{d} \xi_i X_i \right\}.$$

More generally, for a probability measure μ on S^d we may define

$$\varrho_\mu^\infty(X) = \int_{S^d} \varrho_\xi^\infty(X) d\mu(\xi).$$

Theorem (Ekeland, Galichon, Henry, 2009)

Assume that ϱ is a convex, law invariant risk measure on $\mathbb{R}^d$ which extends continuously from $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ to $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, for some $p < \infty$. Then there is a function $v : \mathcal{P} \to [0, \infty]$ such that

$$\varrho(X) = \max_{F \in \mathcal{P}} \{\varrho_F(X) - v(F)\}$$

Theorem (Ekeland, S., 2011)

Assume that $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R}$ is a convex, law invariant risk measure on $\mathbb{R}^d$. Then there is a function $v : [0, 1] \times \mathcal{P} \times \mathcal{M}_+^1(S^d) \to [0, \infty]$ such that

$$\varrho(X) = \max_{(\alpha, F, \mu) \in [0,1] \times \mathcal{P} \times \mathcal{M}_+^1(S^d)} \left\{\alpha \varrho_F(X) + (1 - \alpha)\varrho_\mu^\infty(X) - v(\alpha, F, \mu)\right\}$$

The law invariant risk measure ϱ is coherent if and only if v can be chosen to take only values in $\{0, \infty\}$.

Definition (Ekeland, Galichon, Henry, 2009)

A convex, law invariant risk measure $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R}$ is *strongly coherent* if

$$\varrho(X) + \varrho(Y) = \sup \left\{ \varrho(\tilde{X} + \tilde{Y}) : \tilde{X} \sim X, \tilde{Y} \sim Y \right\}.$$

Theorem (Ekeland, Galichon, Henry, 2009)

Let ϱ be a convex, law invariant risk measure which extends continuously from $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ to $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, for some $p < \infty$. Then ϱ is strongly coherent if and only if there is some $F \in \mathcal{P}$ such that

$$\varrho(X) = \alpha \varrho_F(X).$$

Theorem (Ekeland, S., 2011)

A convex, law invariant risk measure $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R}$ is strongly coherent if and only if there is some $F \in \mathcal{P}, \mu \in \mathcal{M}_+(S^d)$ and $\alpha \in [0, 1]$ such that

$$\varrho(X) = \alpha \varrho_F(X) + (1 - \alpha) \varrho_\mu^\infty(X).$$