On the Law of Large Numbers for random set valued mappings

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Outline

- Joint work with Georg Pflug
- LLN for random set valued mappings
  - LLN: integrably bounded case
  - LLN: unbounded case
- LSN: Concentration inequalities for random mappings
- LLN for nonintegrable random variable: Open problem
- References

Joint work with Georg Pflug: SBB method

Norkin V.I., Pflug G.Ch. and Ruszczyński A. A branch and bound method for stochastic global optimization, Math. Progr., 1998, V. 83, 425-450.

- Idea: Stochastic bounds in the B&B method
- Interchange relaxation: $\min \mathbb{E} \geq \mathbb{E} \min$
- Multiple independent observations for sharpening bounds
  Similar idea: Marco Campi et al. (2006, 2008)

LLN for random sets and set valued mappings

- $S(\xi, x)$ is a random set valued mapping
  with expectation $\mathbb{E}_\xi S(\xi, x)$
- $\{\xi_i, i = 1, 2, ..., \nu\}$ iid r.v.
- LLN for for random sets (x is fixed):
  $S^\nu(x) := \frac{1}{\nu} \sum_{i=1}^{\nu} S(\xi_i, x) \overset{set\ conv.\ a.s.}{\longrightarrow} \mathbb{E}_\xi \text{co} S(\xi, x)$
  Artstein and Vitale (1975), Artstein and Hart (1981)
- LLN for set valued mapping:
  $S^\nu(\cdot) := \frac{1}{\nu} \sum_{i=1}^{\nu} S(\xi_i, \cdot) \overset{?}{\longrightarrow} \mathbb{E}_\xi \text{co} S(\xi, \cdot)$

LLN: Compact valued integrably bounded mappings

- **Assumption:** $\sup_x \|S(\xi, x)\|$ is integrable.

- **Theorem 1** ("Uniform" LLN, Shapiro and Xu (2007)).
  $\forall\ \varepsilon > 0$ a.s.
  $\lim_{\nu\to\infty} \max_{x\in X}$ Dev $(S^\nu(x), \mathbb{E}\text{co}S(\xi, \text{Ball}_\varepsilon(x)) = 0,$
  $\lim_{\nu\to\infty} \max_{x\in X}$ Dev $(\mathbb{E}\text{co}S(\xi, x), S^\nu(\text{Ball}_\varepsilon(x)) = 0.$

- **Theorem 2** (graphical LLN):
  gph$\frac{1}{\nu}\sum_{i=1}^{\nu} S(\xi_i, \cdot) \longrightarrow$ gph$\mathbb{E}_\xi\text{co}S(\xi, \cdot)$ a.s.

- **Theorem 3.** Theorems 1 and 2 are equivalent.

LLN: Integrable unbounded closed valued mappings

- **Theorem 4** (epi-LLN, Attouch and Wets (1990))
  Assume $s(\xi, x)$ is a random lsc function bounded from below. Then
  gph$\frac{1}{\nu}\sum_{i=1}^{\nu}\left(s(\xi_i,\cdot)+I\!R_+\right) \longrightarrow \left(\mathbb{E}_\xi\left(s(\xi,\cdot)+I\!R_+\right)\right.$
- **Theorem 5** (graphical LLN for sum $S+K$ of bounded S and unbounded K random osc mappings).
  Assume
  - $S(\xi,\cdot)$ is compact valued and integrably bounded.
  - $K(\xi,\cdot)$ is outer semi-continuous and convex valued;
  - $\sup_x \inf_{y\in K(\xi,x)} \|y\|$ is integrable;
  - $K(\xi,x) \in \mathbb{E}K(\xi,x)\ \forall\, \xi,\ x$.

  Then
  ghp$\frac{1}{\nu}\sum_{i=1}^{\nu}(S(\xi,\cdot)+K(\xi,\cdot)) \to$ cl-gph$\mathbb{E}(coS(\xi,\cdot)+K(\xi,\cdot))$

LSN: Concentration inequalities for random mappings

- LSN: Concentration of sample average around mean
  - Bounded random variables: Hoeffding (1963)
  - Random functions: Talagrand (1996)
  - Random vectors: Nemirovski (2004, 2008)
  - Random sets: Artstein (1984)

- **Theorem 6** (LSN: concentration of random graphs)
  Assume $S(\xi, \cdot)$ is discretely distributed and bounded by $\|S(X)\|$, then for sample average S^ν holds:
  $$\text{Prob}\{\sqrt{\nu}\ Dist(\text{gph}S^\nu, \text{gph}\mathbb{E}S) > (1+t)\|S(X)\|\} \leq$$
  $$\leq \text{Prob}\{\sqrt{\nu}\ \sup_x(S^\nu(x), \mathbb{E}S(\xi, x)) > (1+t)\|S(X)\|\} \leq$$
  $$\leq \exp\left[-t^2/4\right].$$

LLN for nonintegrable random variables

Exercise. $\{\xi_i\}$ are independent uniformly distributed on [-1,+1].

$$\frac{1}{\nu}\sum_{i=1}^{\nu}\frac{1}{\xi_i} \quad \xrightarrow{?} \quad \textbf{?}$$

Random variable $S(\xi) = 1/\xi$ is not integrable.

Theorem (Chow and Robbins (1961)). For any nonintegrable random variable $S(\xi)$ there is no normalizing sequence m_ν such that

$$\lim_{\nu} \frac{1}{m_\nu}\sum_{i=1}^{\nu} S(\xi_i) = C,$$

where $\{\xi_i\}$ are iid and $0 < C < \infty$, so no LLN can hold.

Conjecture

Numerical experiments give the basis for the following conjecture. Let ξ is uniform in [-1,+1], $\alpha > 0$,

$$S(\xi) = \begin{cases} \mathrm{sgn}(\xi)/|\xi|^{1+\alpha}, & \xi \neq 0, \\ 0, & \xi = 0, \end{cases}$$

Conjecture. Normalized sums

$$S^{\nu} = \frac{1}{\nu^{1+\alpha}} \sum_{i=1}^{\nu} S(\xi_i)$$

converge as $\nu \longrightarrow \infty$ to some fixed (non normal) universal distribution concentrated around zero.

Summary of references

Random sets
Artstein and Vitale (1975)
Artstein and Hart (1981)
Cressie (1979)
Weil (1982)

Concentration
Artstein (1981)
Nemirovski (2004)
Nemirovski and Juditski (2008)

Random mappings
Attouch and Wets (1990)
Shapiro and Xu (2007)
Terán (2008)
Shapiro (2003, 2009)

Ledoux and Talagrand (1991)
Talagrand (1996)
Ledoux (2001)

Thank you for your attention!